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# THE CHROMATIC EXT GROUPS $\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*, M_2^1)$

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Abstract. We compute a certain Ext group related to the chromatic spectral sequence for $T(m)$, the spectrum whose $BP$-homology is $BP_*[t_1, \cdots, t_m]$ for each $m \geq 3$. The answer we get displays a kind of periodicity not seen in the corresponding computation for the sphere spectrum.

## 1. Introduction

Let $BP$ be the Brown-Peterson spectrum for a fixed prime $p$. In [Rav86, §6.5], the third author introduced the spectrum $T(m)$ which has $BP_*$-homology

$$BP_*(T(m)) = BP_*[t_1, \cdots\cdots, t_m],$$

and is homotopy equivalent to $BP$ below dimension $2p^{m+1} - 3$. $T(m)$ is a retract of the localization at the prime $p$ of the Thom spectrum $X(i)$ associated with the map

$$\Omega SU(i) \to \Omega SU = BU$$

for any $i$ satisfying $p^m \leq i < p^{m+1}$. The $X(i)$ figured in the proof of the nilpotence theorem of [DHS88]. Notice that $T(0)$ is just the sphere spectrum. The $T(m)$ are used in the method of infinite descent, the third author's program for computing the stable homotopy groups of spheres described in [Rav86, Chapter 7] and in [Rav]. In it one gets at the Adams-Novikov $E_2$-term

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*)$$

for the sphere spectrum by computing the one for $T(m)$,

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m)))$$

by downward induction on $m$. In given range of the dimensions, the latter group is isomorphic to $BP_*$ (concentrated in cohomological degree 0) for $m$ sufficiently large. One can pass from $m$ to $m-1$ by a certain collection of spectral sequences.

There is a change-of-ring isomorphism [Rav86, 7.1.3] giving

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))) \cong \mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*),$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \ldots, t_m) = BP_*[t_{m+1}, t_{m+2}, \ldots].$$

In particular $\Gamma(1) = BP_*(BP)$ by definition. This Ext group stabilizes in a certain sense as $m$ gets large, which is studied in [Rav00]. To get at its structure, we

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can use the chromatic method introduced in [MRW77] and described in [Rav86, Chapter 5].

Define the chromatic module $M_n^s$ by

$$M_n^s = v_{n+s}^{-1} BP_*/(p,\, v_1,\, \cdots,\, v_{n-1},\, v_n^\infty,\, \cdots,\, v_{n+s-1}^\infty).$$

Hereafter we will abbreviate $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ as $\mathrm{Ext}_{\Gamma(m+1)}(M)$ for simplicity. Consider the chromatic spectral sequence converging to $\mathrm{Ext}_{\Gamma(m+1)}(BP_*/I_n)$ with

$$E_1^{s,t} = \mathrm{Ext}^t_{\Gamma(m+1)}(M_n^s) \qquad \text{and} \qquad d_r : E_r^{s,t} \to E_r^{s+r,t-r+1},$$

where $E_\infty^{s,t}$ is a subquotient of $\mathrm{Ext}^{s+t}_{\Gamma(m+1)}(BP_*/I_n)$. Shimomura calls this $E_1^{s,t}$ the **general chromatic $E_1$-term**. In this paper we will determine the module structure of

$$\mathrm{Ext}^0_{\Gamma(m+1)}(M_2^1).$$

The analogous result for $m = 0$ was obtained long ago by Miller and Wilson in [MW76], and is as follows.

**Theorem 1.1** (Miller and Wilson). *As a $k(2)_*$-module, $\mathrm{Ext}^0_{\Gamma(1)}(M_2^1)$ is the direct sum of*

1. *the cyclic submodules freely generated by $x_k^s/v_2^{a(k)}$ for $k \geq 0$ and $s \in \mathbf{Z} - p\mathbf{Z}$, where*

$$\begin{aligned} x_0 &= v_3, \\ x_1 &= v_3^p - v_2^p v_3^{-1} v_4, \end{aligned}$$

*and for $k \geq 2$*

$$x_k = \begin{cases} x_{k-1}^p & \text{for } k \text{ even}, \\ x_{k-1}^p - v_2^{(p^{k-1}-1)(p^3-1)/(p^2-1)} v_3^{p^k - p^{k-1}+1} & \text{for } k \text{ odd}, \end{cases}$$

*and*

$$\begin{aligned} a(0) &= 1, \\ a(1) &= p, \end{aligned}$$

*and for $k \geq 2$*

$$a(k) = \begin{cases} pa(k-1) & \text{for } k \text{ even}, \\ pa(k-1) + p - 1 & \text{for } k \text{ odd}; \end{cases}$$

*and*

2. *$K(2)_*/k(2)_*$, generated by $1/v_2^j$ for $j \geq 1$.*

Before this result was proved, the naive conjecture about this group would have had the exponents $a(k)$ being $p^k$ for all $k \geq 0$. It was clear that

$$\frac{v_3^{sp^k}}{v_2^{p^k}} \in \mathrm{Ext}^0_{\Gamma(1)}(M_2^1),$$

but the existence of "deeper" elements such as

$$\frac{x_3}{v_1^{a(3)}} = \frac{v_3^{p^3} - v_2^{p^3} v_3^{-p^2} v_4^{p^2} - v_2^{p^3-1} v_3^{p^3-p^2+1}}{v_2^{p^3+p-1}}$$

came as a surprise, as did the fact that the limiting value (as $k \to \infty$) of $a(k)/p^k$ is $(p^2 + p + 1)/(p^2 + p)$ instead of 1.

Our result (Theorem 6.4 below) has the same form as Theorem 1.1 but with $x_k$ and $a(k)$ replaced by $\widehat{x}_k$ ((4.2) and (6.2)) and $\widehat{a}(k)$ ((4.1) and (6.1)), and with $k(2)_*$ replaced by a larger ring $\widehat{k}(2)_*$ defined in (1.2). In order to avoid the excessive appearance of the index $m$, we will use the following notation:

$$(1.2)\qquad \begin{cases} \widehat{v}_i = v_{m+i}, & \widehat{t}_i = t_{m+i}, \\ \widehat{K}(n)_* = K(n)_*[v_{n+1},\ldots,v_{n+m}], & \widehat{h}_{i,j} = h_{m+i,j}, \\ \widehat{k}(n)_* = k(n)_*[v_{n+1},\ldots,v_{n+m}], \quad \text{and} & \omega = p^m. \end{cases}$$

In §§7–10 we will define elements $\widehat{x}_k \in v_3^{-1}BP_*/I_2$ and integers $\widehat{a}(k)$ whenever the condition $3 < 2(p-1)(m+1)/p$ of Theorem 2.1 is satisfied, and compute the reduced right unit $d_0$ on $\widehat{x}_k$. These will depend on $m$.

Using these computations, we can obtain the structure of the target Ext group. In particular, for large $m$ we have

**Theorem 1.3.** *Assume that $p = 2$ and $m \geq 6$ or that $p > 2$ and $m \geq 5$. As a $\widehat{k}(2)_*$-module, $\mathrm{Ext}_{\Gamma(m+1)}(M_2^1)$ is the direct sum of*

1. *the cyclic submodules generated by $\widehat{x}_k^s/v_2^{\widehat{a}(k)}$ for $k \geq 0$ and $p \nmid s > 0$, where*

$$\widehat{x}_k = \begin{cases} \widehat{v}_3 & \text{for } k=0, \\ \widehat{x}_2^p - v_2^{p^3} v_3^{-p^2} \widehat{v}_4^{p^2} - v_2^{p^3-1} v_3^{(p\omega-1)p^2\widehat{v}_3} & \text{for } k=3, \\ \widehat{x}_5^p + v_2^{(p+1)p^5} v_3^{-p^5} W & \text{for } k=6, \\ \widehat{x}_8^p - v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X^{p^2} - v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6} \widehat{x}_6 & \\ & \text{for } k=9, \\ \widehat{x}_{k-1}^p & \\ & \text{for } 3 \nmid k \text{ and } k \leq 8, \\ \widehat{x}_{k-1}^p + v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1}\left(\widehat{x}_{k-4} - \widehat{x}_{k-5}^p\right) & \text{for } k \geq 10; \end{cases}$$

*(see Lemma 5.2 and Proposition 5.9 for definitions of $W$ and $X$) and*

$$\widehat{a}(k) = \begin{cases} p^k & \text{for } 0 \leq k \leq 2, \\ (p+1)p^{k-1} & \text{for } 3 \leq k \leq 5, \\ (p^2+p+1)p^{k-2} & \text{for } 6 \leq k \leq 8, \\ p^k + p^{k-1} + p^{k-2} + \widehat{a}(k-4) & \text{for } k \geq 9; \end{cases}$$

*and*

2. *$\widehat{K}(2)_*/\widehat{k}(2)_*$, generated by $1/v_2^j$ for $j \geq 1$.*

$\square$

This is a part of our main result (Theorem 6.4). In fact, we have determined the structure of $\mathrm{Ext}_{\Gamma(m+1)}(M_2^1)$ for $p = 2$ and $m \geq 3$, or $p > 2$ and $m \geq 2$.

At the JAMI conference held at Johns Hopkins University in March 2000, Shimomura reported that he had extended our result and obtained the structure of

$$\mathrm{Ext}^0_{\Gamma(m+1)}(M_{n-1}^1) \quad \text{for } m \geq n^2 - n - 1.$$

For $n = 3$, this result coincides with our Theorem 1.3. Recently he also determined the structure of higher Ext groups ([Shi]).

In the above case, the asymptotic behavior of the exponents is given by

$$\lim_{k\to\infty} \frac{\widehat{a}(k)}{p^k} = \frac{p^4+p^3+p^2}{p^4-1},$$

a slightly larger value than for the case $m = 0$. In addition, there is a new form of periodicity in our statement with no precedent in Theorem 1.1; namely, for all $k \geq 9$ we have

$$\widehat{x}_k - \widehat{x}_{k-1}^p = -v_2^{p^k+p^{k-1}+p^{k-2}} v_3^{p^{m+k}-p^{k-1}-p^{k-2}-p^{k-3}} \widehat{x}_{k-4}^{p-1} \left(\widehat{x}_{k-4} - \widehat{x}_{k-5}^p\right),$$

and

$$\widehat{a}(k) = p^k + p^{k-1} + p^{k-2} + \widehat{a}(k-4).$$

(For lower $m$ the period is 6 instead of 4.) A similar result for the chromatic module $M_1^1$ is obtained in [NR] or [KS], in which the period is 3 instead of 4.

## 2. Preliminaries

For a $\Gamma(m+1)$-comodule $M$, consider the cobar complex $\left\{C^n_{\Gamma(m+1)}(M), d_n\right\}_{n\geq 0}$, where

$$C^n_{\Gamma(m+1)}(M) = \underbrace{\Gamma(m+1) \otimes_{BP_*} \cdots\cdots \otimes_{BP_*} \Gamma(m+1)}_{n \text{ factors}} \otimes_{BP_*} M,$$

and

$$d_n : C^n_{\Gamma(m+1)}(M) \to C^{n+1}_{\Gamma(m+1)}(M).$$

Then $\mathrm{Ext}_{\Gamma(m+1)}(M)$ is given as the cohomology group of this cobar complex. By the change-of-ring isomorphism ([Rav86, Theorem 6.1.1]), we have

$$\begin{aligned} \mathrm{Ext}_{\Gamma(m+1)}\left(M_n^0\right) &\cong \mathrm{Ext}_{\Gamma(1)}\left(M_n^0 \otimes BP_*(T(m))\right) \\ &\cong \mathrm{Ext}_{\Sigma(n)}\left(K(n)_*, K(n)_*(T(m))\right). \end{aligned}$$

This object is already known by [Rav86, Corollary 6.5.6]:

**Theorem 2.1.** *If $n < 2(p-1)(m+1)/p$, then*

$$\mathrm{Ext}_{\Gamma(m+1)}\left(M_n^0\right) = \widehat{K}(n)_* \otimes E\left(\widehat{h}_{i,j} : 1 \leq i \leq n, 0 \leq j \leq n-1\right),$$

*where each $\widehat{h}_{i,j}$ lies in* $\mathrm{Ext}^1$. □

In particular, we have

$$\mathrm{Ext}_{\Gamma(m+1)}\left(M_3^0\right) = \widehat{K}(3)_* \otimes E\left(\widehat{h}_{i,j} : 1 \leq i \leq 3,\ 0 \leq j \leq\right). \tag{2.2}$$

From this information we can get the structure of $\mathrm{Ext}_{\Gamma(m+1)}\left(M_2^1\right)$ using the Bockstein spectral sequence.

**Lemma 2.3** (cf. [MRW77], Remark 3.11)**.** *Assume that there exists a $\widehat{k}(2)_*$-submodule $B^t$ of $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right)$ for each $t < N$, such that the following sequence is* *exact:*

$$\begin{array}{l}
\mathrm{Ext}^0_{\Gamma(m+1)}\left(M^0_3\right) \xrightarrow{1/v_2} B^0 \xrightarrow{v_2} B^0 \xrightarrow{\delta} \mathrm{Ext}^1_{\Gamma(m+1)}\left(M^0_3\right) \xrightarrow{1/v_2} B^1 \xrightarrow{v_2} \cdots \\
\cdots \longrightarrow B^{N-1} \xrightarrow{v_2} B^{N-1} \xrightarrow{\delta} \mathrm{Ext}^N_{\Gamma(m+1)}\left(M^0_3\right),
\end{array}$$

*where $\delta$ is the restriction of the coboundary map*

$$\delta : \mathrm{Ext}^t_{\Gamma(m+1)}\left(M^1_2\right) \to \mathrm{Ext}^{t+1}_{\Gamma(m+1)}\left(M^0_3\right).$$

*Then the inclusion map $i_t : B^t \to \mathrm{Ext}^t_{\Gamma(m+1)}\left(M^1_2\right)$ is an isomorphism between $\widehat{k}(2)_*$-modules for each $t < N$.*

*Proof.* Because $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M^1_2\right)$ is a $v_2$-torsion module, we can define filtrations on $B^t$ and $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M^1_2\right)$ by

$$P_t(j) = \left\{x \in B^t \colon v_2^j x = 0\right\} \quad \text{and} \quad Q_t(j) = \left\{x \in \mathrm{Ext}^t_{\Gamma(m+1)}\left(M^1_2\right) : v_2^j x = 0\right\}.$$

Assume that the inclusion $i_k$ is an isomorphism for $k \le t-1$ (the $t = 0$ case is obvious), and consider the following commutative ladder diagram:

$$\begin{array}{ccccccc}
\mathrm{Ext}^t_{\Gamma(m+1)}\left(M^1_2\right) & \xrightarrow{1/v_2} & P_t(j) & \xrightarrow{v_2} & P_t(j-1) & \xrightarrow{\delta} & \mathrm{Ext}^{t+1}_{\Gamma(m+1)}\left(M^1_2\right) \\
\| & & \downarrow i_t & & \downarrow i_t & & \| \\
\mathrm{Ext}^t_{\Gamma(m+1)}\left(M^1_2\right) & \xrightarrow{1/v_2} & Q_t(j) & \xrightarrow{v_2} & Q_t(j-1) & \xrightarrow{\delta} & \mathrm{Ext}^{t+1}_{\Gamma(m+1)}\left(M^1_2\right).
\end{array}$$

Because the inclusion from $B^{t-1}$ to $\mathrm{Ext}^{t-1}_{\Gamma(m+1)}\left(M^2_1\right)$ is an isomorphism by assumption, we can show that $P_t(j)$ is isomorphic to $Q_t(j)$ $(j \ge 1)$ using the Five Lemma and induction on $j$. $\square$

In (4.2) and (6.2) we will define elements $\widehat{x}_k$ of $v_3^{-1}BP_*/I_2$, which is congruent to $\widehat{v}_3^{sp^k}$ modulo $(v_2)$, and integers $\widehat{a}(k)$ in (4.1) and (6.1) such that each $\widehat{x}^s_k/v_2^\ell$ is a cycle of $\mathrm{Ext}^0_{\Gamma(m+1)}\left(M^1_2\right)$ for all $1 \le \ell \le \widehat{a}(k)$.

Then the structure of $B^0$ is expressed as follows:

**Lemma 2.4.** *As a $\widehat{k}(2)_*$-module,*

$$B^0 = \widehat{k}(2)_* \left\{ \frac{\widehat{x}^s_k}{v_2^{\widehat{a}(k)}} \colon \ k \ge 0 \text{ and } p \nmid s > 0 \right\} \oplus \widehat{K}(2)_*/\widehat{k}(2)_*,$$

*satisfies the condition of Lemma 2.3 if the set*

$$\left\{ \delta\left(\frac{\widehat{x}^s_k}{v_2^{\widehat{a}(k)}}\right) : \ k \ge 0 \text{ and } p \nmid s > 0 \right\} \subset \mathrm{Ext}^1_{\Gamma(m+1)}\left(M^0_3\right)$$

*is linearly independent, where $\delta$ is the coboundary map in Lemma 2.3. Hence it is isomorphic to* $\mathrm{Ext}^0_{\Gamma(m+1)}\left(M^1_2\right)$.

*Proof.* We will show that the following sequence is exact:

$$0 \longrightarrow \mathrm{Ext}^0_{\Gamma(m+1)}\left(M^0_3\right) \xrightarrow{1/v_2} B^0 \xrightarrow{v_2} B^0 \xrightarrow{\delta} \mathrm{Ext}^1_{\Gamma(m+1)}\left(M^0_3\right).$$

The only part of this that is not obvious is that Ker $\delta \subset$ Im $v_2$. To show this, separate the $\mathbf{Z}/(p)$-basis of $B^0$ into two parts,

$$\begin{aligned} A &= \left\{ \frac{\widehat{x}_k^s}{v_2^{\widehat{a}(k)}} : \; k \geq 0 \text{ and } p \nmid s > 0 \right\}, \\ B &= \left\{ \frac{\widehat{x}_k^s}{v_2^{\ell}} : \; k \geq 0, \, p \nmid s > 0, \text{ and } 1 \leq \ell < \widehat{a}(k) \right\} \cup \left\{ v_2^{-j} : j > 0 \right\}. \end{aligned}$$

Then it is obvious that $\delta(x_\lambda) \neq 0 \in \mathrm{Ext}^1_{\Gamma(m+1)}\left(M^0_3\right)$ for $x_\lambda \in A$, and that $\delta(y_\mu) = 0 \in \mathrm{Ext}^1_{\Gamma(m+1)}\left(M^0_3\right)$ for $y_\mu \in B$. Thus for any element $z = \sum_\lambda a_\lambda x_\lambda + \sum_\mu b_\mu y_\mu$ of $B^0$ $(a_\lambda, b_\mu \in \mathbf{Z}/(p))$, we have $\delta(z) = \sum_\lambda a_\lambda \delta(x_\lambda)$. The condition implies that all $a_\lambda$ are zero when $\delta(z) = 0$, and so $v_2 \sum_\mu b_\mu y_\mu / v_2 = z$. This completes the proof. □

## 3. Elementary calculations

In this section we will introduce elements $\widehat{w}_4$ and $\widehat{w}_5$, which we need to define our $\widehat{x}_k$. First we recall the right unit on $\widehat{v}_i$.

**Lemma 3.1.** *Assume that $p \geq 2$ and $m \geq 1$. In $\Gamma(m+1)/(p, v_1)$ the right unit map $\eta_R : \widehat{A} \to \widehat{\Gamma}$ on the element $\widehat{v}_i$ is given by*

$$\begin{aligned} \eta(\widehat{v}_i) &= \widehat{v}_i && \text{for } i \leq 2, \\ \eta(\widehat{v}_3) &= \widehat{v}_3 + v_2 \widehat{t}_1^{p^2} - v_2^{p\omega} \widehat{t}_1, \\ \eta(\widehat{v}_4) &= \widehat{v}_4 + v_3 \widehat{t}_1^{p^3} + v_2 \widehat{t}_2^{p^2} - v_3^{p\omega} \widehat{t}_1 - v_2^{p^2\omega} \widehat{t}_2, \\ \eta(\widehat{v}_5) &\equiv \widehat{v}_5 + v_4 \widehat{t}_1^{p^4} + v_3 \widehat{t}_2^{p^3} + v_2 \widehat{t}_3^{p^2} \\ & \quad - v_4^{p\omega} \widehat{t}_1 - v_3^{p^2\omega} \widehat{t}_2 - v_2^{p+1} \widehat{v}_3^{p(p-1)} \widehat{t}_1^{p^3} \quad \bmod \left( v_2^{2p+1} \right) \\ & \quad (\textit{add } v_2^4 \widehat{t}_1^{17} \textit{ when } (p, m) = (2, 1)). \end{aligned}$$

*Moreover, when $p \geq 2$ and $m \geq 2$, we have*

$$\eta(\widehat{v}_6) \equiv \widehat{v}_6 + v_5 \widehat{t}_1^{p^5} + v_4 \widehat{t}_2^{p^4} + v_3 \widehat{t}_3^{p^3} - v_5^{p\omega} \widehat{t}_1 - v_4^{p^2\omega} \widehat{t}_2 - v_3^{p^3\omega} \widehat{t}_3 \quad \bmod (v_2).$$

Define elements $\widehat{w}_i$ for $4 \leq i \leq 5$ by

$$\begin{aligned} \widehat{w}_4 &= v_3^{-1} \widehat{v}_4, \\ \widehat{w}_5 &= v_3^{-1} \left( \widehat{v}_5 - v_4 \widehat{w}_4^p + v_2^{p+1} \widehat{v}_3^{(p-1)p} \widehat{w}_4 \right). \end{aligned}$$

Then we have the following lemma.

**Lemma 3.2.** *For any prime $p$, we have*

$$\begin{aligned} d(\widehat{w}_4) &= \widehat{t}_1^{p^3} + v_2 v_3^{-1}\widehat{t}_2^{p^2} - v_3^{p\omega-1}\widehat{t}_1 - v_2^{p^2\omega} v_3^{-1}\widehat{t}_2 \qquad \text{for } m \geq 1,\\ d(\widehat{w}_5) &\equiv \widehat{t}_2^{p^3} + v_2 v_3^{-1}\widehat{t}_3^{p^2} - v_3^{-1}v_4^{\omega p}\widehat{t}_1 - v_3^{p^2\omega-1}\widehat{t}_2 - v_2^p v_3^{-p-1}v_4\widehat{t}_2^{p^3} + v_3^{p^2\omega-p-1}v_4\widehat{t}_1^p\\ &\quad + v_2^{p+2}v_3^{-2}\widehat{v}_3^{(p-1)p}\widehat{t}_2^{p^2} - v_2^{p+1}v_3^{p\omega-2}\widehat{v}_3^{(p-1)p}\widehat{t}_1 \mod \left(v_2^{2p+1}\right) \quad \text{for } m \geq 2\\ &\quad (\text{add } v_2^4 v_3^{-1}\widehat{t}_1^{17} \text{ when } (p, m) = (2, 1)). \end{aligned}$$

*Proof.* $d(\widehat{w}_4)$ is easily computed using Lemma 3.1. For $\widehat{w}_5$ we find that

$$d(\widehat{v}_5) \equiv v_4\widehat{t}_1^{p^4} + v_3\widehat{t}_2^{p^3} + v_2\widehat{t}_3^{p^2} - v_4^{p\omega}\widehat{t}_1 - v_3^{p^2\omega}\widehat{t}_2 - v_2^{p+1}\widehat{v}_3^{p(p-1)}\widehat{t}_1^{p^3} \mod \left(v_2^{2p+1}\right)$$
$$(\text{add } v_2^4\widehat{t}_1^{17} \text{ for } (p, m) = (2, 1)).$$

We can read off the fact that $d(v_4) = 0$ for $m \geq 2$ and $d(\widehat{v}_3) \equiv 0$ modulo $(v_2)$ from Lemma 3.1. We have

$$\begin{aligned} d\left(-v_4\widehat{w}_4^p\right) &\equiv -v_4\left(\widehat{t}_1^{p^4} + v_2^p v_3^{-p}\widehat{t}_2^{p^3} - v_3^{(p\omega-1)p}\widehat{t}_1^p\right) &&\mod \left(v_2^{p^3\omega}\right),\\ d\left(v_2^{p+1}\widehat{v}_3^{p(p-1)}\widehat{w}_4\right) &\equiv v_2^{p+1}\widehat{v}_3^{p(p-1)}\left(\widehat{t}_1^{p^3} + v_2 v_3^{-1}\widehat{t}_2^{p^2} - v_3^{p\omega-1}\widehat{t}_1\right) &&\mod \left(v_2^{2p+1}\right). \end{aligned}$$

Summing these congruences and multiplying by $v_3^{-1}$ gives $d(\widehat{w}_5)$. □

## 4. $d(\widehat{x}_k)$ for $0 \leq k \leq 5$

For all $p$ and $m$ we can construct $\widehat{x}_k$ $(0 \leq k \leq 5)$ and compute differentials on these one at a time (Lemma 4.3). Define integers $\widehat{a}(k)$ (as in Theorem 1.3) and $\widehat{b}(k)$ for $0 \leq k \leq 5$ by

$$\widehat{a}(k) = \begin{cases} p^k & \text{if } 0 \leq k \leq 2,\\ (p+1)p^{k-1} & \text{if } 3 \leq k \leq 5, \end{cases}$$
$$\widehat{b}(k) = \begin{cases} 0 & \text{if } 0 \leq k \leq 2,\\ -p^{k-1} & \text{if } 3 \leq k \leq 5. \end{cases} \tag{4.1}$$

Define elements (as in Theorem 1.3) $\widehat{x}_k$ for $0 \leq k \leq 5$ by

$$\begin{cases} \widehat{x}_0 = \widehat{v}_3,\\ \widehat{x}_3 = \widehat{x}_2^p - v_2^{p^3}\widehat{w}_4^{p^2} - v_2^{p^3-1}v_3^{(p\omega-1)p^2}\widehat{x}_0,\\ \widehat{x}_k = \widehat{x}_{k-1}^p \qquad \text{for } k \not\equiv 0 \bmod (3). \end{cases} \tag{4.2}$$

**Lemma 4.3.** *Assume that $p \geq 2$ and $m \geq 2$. In the module $M_2^1 \otimes \Gamma(m+1)$ we have*

$$d(\widehat{x}_k) \equiv v_2^{\widehat{a}(k)}v_3^{\widehat{b}(k)} \begin{cases} \widehat{t}_1^{p^{k+2}} & \mod \left(v_2^{p^{k+1}\omega}\right) & \text{for } 0 \leq k \leq 2,\\ -\widehat{t}_2^{p^{k+1}} & \mod \left(v_2^{p^{k-3}(p^3+p\omega-1)}\right) & \text{for } 3 \leq k \leq 5. \end{cases}$$

*In particular, these congruences hold modulo $(v_2^{1+\widehat{a}(k)})$.*

*Proof.* We obtain $d(\widehat{x}_k)$ for $0 \leq k \leq 2$ from $\eta_R(\widehat{v}_3)$ (Lemma 3.1). For $d(\widehat{x}_3)$, we have

$$\text{(4.4)}\quad \begin{cases} d(\widehat{x}_2^p) \equiv v_2^{p^3}\widehat{t}_1^{p^5} & \bmod \left(v_2^{p^4\omega}\right), \\ d\left(-v_2^{p^3}\widehat{w}_4^{p^2}\right) \equiv -v_2^{p^3}\left(\widehat{t}_1^{p^5}+v_2^{p^2}v_3^{-p^2}\widehat{t}_2^{p^4}-v_3^{(p\omega-1)p^2}\widehat{t}_1^{p^2}\right) & \\ \qquad\qquad & \bmod \left(v_2^{(p\omega+1)p^3}\right), \\ d\left(-v_2^{p^3-1}v_3^{(p\omega-1)p^2}\widehat{x}_0\right) = -v_2^{p^3}v_3^{(p\omega-1)p^2}\left(\widehat{t}_1^{p^2}-v_2^{p\omega-1}\widehat{t}_1\right). & \end{cases}$$

Notice that

$$p^3+p\omega-1<\min\left\{p^4\omega,(p\omega+1)p^3\right\}=p^4\omega$$

for all $p$ and $m$. Summing congruences in (4.4), we have

$$d(\widehat{x}_3)\equiv -v_2^{p^3+p^2}v_3^{-p^2}\widehat{t}_2^{p^4}+v_2^{p^3+p\omega-1}v_3^{(p\omega-1)p^2}\widehat{t}_1 \qquad \bmod \left(v_2^{p^4\omega}\right).$$

Noticing that the inequality $p^3+p\omega-1>p^3+p^2$ holds only if $m\geq 2$, we find that

$$d(\widehat{x}_3)\equiv \begin{cases} v_2^{p^3+p^2-1}v_3^{(p^2-1)p^2}\widehat{t}_1 & \bmod \left(v_2^{p^3+p^2}\right) & \text{for } m=1, \\ -v_2^{p^3+p^2}v_3^{-p^2}\widehat{t}_2^{p^4} & \bmod \left(v_2^{p^3+p\omega-1}\right) & \text{for } m\geq 2. \end{cases}$$

By assumption, we may consider only the case that $m\geq 2$, and set $\widehat{a}(3)=p^3+p^2$. The formulas for $4\leq k\leq 5$ are obvious. □

To define $\widehat{x}_k$ for higher $k$, we will prepare some lemmas in the next section. The definitions of $\widehat{x}_k$ ($k\geq 6$) and computations of the chromatic differential $d_0$ on $\widehat{x}_k$ are separated into 4 sections (§§7–10) according to the value of $m$. The results are stated in section 6.

## 5. Some lemmas

Here we will prove some lemmas for later use. In the rest of this paper we will treat $\mathrm{Ext}^0_{\Gamma(m+1)}(M_2^1)$ whenever the condition in Theorem 2.1,

$$\text{(5.1)}\qquad 3<2(p-1)(m+1)/p,$$

is satisfied. This is equivalent to

$$\begin{cases} m\geq 2 & \text{for } p\geq 3, \\ m\geq 3 & \text{for } p=2. \end{cases}$$

**Lemma 5.2.** *Let*

$$\begin{aligned} W \;=\;& \widehat{w}_5^{p^4}+v_2^{-\widehat{a}(2)}v_3^{-p^4}v_4^{p^5\omega}\widehat{x}_2-v_2^{-\widehat{a}(3)}v_3^{(p^2\omega-1)p^4+p^2}\widehat{x}_3 \\ &- v_2^{p^5-p\widehat{a}(5)}v_3^{-p^4}v_4^{p^4}\widehat{x}_5^p-v_2^{-p\widehat{a}(2)}v_3^{(p^2\omega-p-1)p^4}v_4^{p^4}\widehat{x}_2^p \\ &+ v_2^{(p+1)p^4-\widehat{a}(2)}v_3^{(p\omega-2)p^4}\widehat{v}_3^{(p-1)p^5}\widehat{x}_2. \end{aligned}$$

*Then*

$$\begin{aligned} d(W) \;\equiv\;& \widehat{t}_2^{p^7}+v_2^{p^4}v_3^{-p^4}\widehat{t}_3^{p^6}-v_2^{p^4}v_3^{-p^4}\widehat{v}_3^{(p-1)p^5}d(\widehat{x}_5) \\ &- v_2^{p\omega-p^2-1}v_3^{(p^2\omega-1)p^4+p^3\omega}\widehat{t}_1 \qquad \bmod \left(v_2^{e_1(p,m)}\right), \end{aligned}$$

*where*

$$e_1(p,m) = \begin{cases} (p^3-1)p^2 & \text{if } m=2, \\ (p^4-1)p^2 & \text{if } (p,\, m) = (2,\, 3), \\ (2p^3+p^2-1)p^2 & \text{otherwise.} \end{cases}$$

*Proof.* We find the following congruences:

$$\begin{cases} d\left(\widehat{w}_5^{p^4}\right) & \equiv \ \widehat{t}_2^{p^7} + v_2^{p^4} v_3^{-p^4} \widehat{t}_3^{p^6} - v_3^{-p^4} v_4^{p^5\omega} \widehat{t}_1^{p^4} - v_3^{(p^2\omega-1)p^4} \widehat{t}_2^{p^4} \\ & \quad - v_2^{p^5} v_3^{(-p-1)p^4} v_4^{p^4} \widehat{t}_2^{p^7} + v_3^{(p^2\omega-p-1)p^4} v_4^{p^4} \widehat{t}_1^{p^5} \\ & \quad + v_2^{(p+2)p^4} v_3^{-2p^4} \widehat{v}_3^{(p-1)p^5} \widehat{t}_2^{p^6} - v_2^{(p+1)p^4} v_3^{(p\omega-2)p^4} \widehat{v}_3^{(p-1)p^5} \widehat{t}_1^{p^4} \\ & \qquad\qquad \bmod \left(v_2^{(2p+1)p^4}\right), \\ d\left(v_2^{-\widehat{a}(2)} v_3^{-p^4} v_4^{p^5\omega} \widehat{x}_2\right) & \\ & \equiv \ v_3^{-p^4} v_4^{p^5\omega} \widehat{t}_1^{p^4} \qquad \bmod \left(v_2^{(p\omega-1)p^2}\right), \\ d\left(-v_2^{-\widehat{a}(3)} v_3^{(p^2\omega-1)p^4+p^2} \widehat{x}_3\right) & \\ & \equiv \ v_3^{(p^2\omega-1)p^4} \widehat{t}_2^{p^4} - v_2^{p\omega-p^2-1} v_3^{(p^2\omega-1)p^4+p^3\omega} \widehat{t}_1 \quad \bmod \left(v_2^{(p^2\omega-p-1)p^2}\right), \\ d\left(-v_2^{p^5-p\widehat{a}(5)} v_3^{-p^4} v_4^{p^4} \widehat{x}_5^p\right) & \\ & \equiv \ v_2^{p^5} v_3^{(-p-1)p^4} v_4^{p^4} \widehat{t}_2^{p^7} \qquad \bmod \left(v_2^{(p\omega-1)p^3}\right), \\ d\left(-v_2^{-p\widehat{a}(2)} v_3^{(p^2\omega-p-1)p^4} v_4^{p^4} \widehat{x}_2^p\right) & \\ & \equiv \ -v_3^{(p^2\omega-p-1)p^4} v_4^{p^4} \widehat{t}_1^{p^5} \qquad \bmod \left(v_2^{(p\omega-1)p^3}\right), \\ d\left(v_2^{(p+1)p^4-\widehat{a}(2)} v_3^{(p\omega-2)p^4} \widehat{v}_3^{(p-1)p^5} \widehat{x}_2\right) & \\ & \equiv \ v_2^{(p+1)p^4} v_3^{(p\omega-2)p^4} \widehat{v}_3^{(p-1)p^5} \widehat{t}_1^{p^4} \qquad \bmod \left(v_2^{(2p+1)p^4-p^2}\right). \end{cases}$$

Summing these congruences gives the desired formula. The integer $e_1(p,m)$ is the minimum exponent of $v_2$ in these indeterminacies. □

**Lemma 5.3.** *Let*

$$\begin{aligned} Y \ = \ & -v_3^{p^4\omega+1}\Big(\widehat{w}_4 - v_2^{-p}\widehat{x}_1 + v_2 v_3^{-1-(p^2\omega-1)p^2}\widehat{w}_5^{p^2} + v_2^{1-\widehat{a}(4)} v_3^{p^3-1-(p^2\omega-1)p^2}\widehat{x}_4 \\ & + v_3^{-1-p^4\omega} v_4^{p^3\omega}\widehat{x}_0 + v_2^{1+p^3-\widehat{a}(4)} v_3^{-1-p^4\omega} v_4^{p^2}\widehat{x}_4 - v_2^{1-p} v_3^{-1-p^3} v_4^{p^2}\widehat{x}_1 \\ & + v_2^{(p+1)p^2} v_3^{p^3\omega(1-p)-1-p^2}\widehat{v}_3^{(p-1)p^3}\widehat{x}_0\Big). \end{aligned}$$

*Then*

$$d(Y) \equiv v_3^{p\omega(p^3+1)}\widehat{t}_1 - v_2^{p^2+1}\widehat{t}_3^{p^4} + v_2^{p^2+1}\widehat{v}_3^{(p-1)p^3} d(\widehat{x}_3) + v_2^{p\omega} v_4^{p^3\omega}\widehat{t}_1 \quad \bmod \left(v_2^{e_2(p,m)}\right),$$

*where*

$$e_2(p,m) = \begin{cases} (p-1)(p^3-1) & \text{for } (p,\, m) = (3,\, 2), \\ (2p+1)p^2 & \text{otherwise.} \end{cases}$$

*Proof.* We find the following congruences:

$$\begin{cases} d\left(\widehat{w}_4\right) \equiv \widehat{t}_1^{p^3} + v_2 v_3^{-1} \widehat{t}_2^{p^2} - v_3^{p\omega-1}\widehat{t}_1 & \bmod \left(v_2^{p^2\omega}\right), \\ d\left(-v_2^{-p}\widehat{x}_1\right) \equiv -\widehat{t}_1^{p^3} & \bmod \left(v_2^{(p\omega-1)p}\right), \\ d\left(v_2 v_3^{-1-(p^2\omega-1)p^2}\widehat{w}_5^{p^2}\right) & \\ \quad \equiv v_2 v_3^{-1-(p^2\omega-1)p^2}\Big(\widehat{t}_2^{p^5} + v_2^{p^2} v_3^{-p^2}\widehat{t}_3^{p^4} - v_3^{-p^2} v_4^{p^3\omega}\widehat{t}_1^{p^2} & \\ \qquad - v_3^{(p^2\omega-1)p^2}\widehat{t}_2^{p^2} - v_2^{p^3} v_3^{(-p-1)p^2} v_4^{p^2}\widehat{t}_2^{p^5} + v_3^{(p^2\omega-p-1)p^2} v_4^{p^2}\widehat{t}_1^{p^3} & \\ \qquad + v_2^{(p+2)p^2} v_3^{-2p^2}\widehat{v}_3^{(p-1)p^3}\widehat{t}_2^{p^4} - v_2^{(p+1)p^2} v_3^{(p\omega-2)p^2}\widehat{v}_3^{(p-1)p^3}\widehat{t}_1^{p^2}\Big) & \\ & \bmod \left(v_2^{1+(2p+1)p^2}\right), \\ d\left(v_2^{1-\widehat{a}(4)} v_3^{p^3-1-(p^2\omega-1)p^2}\widehat{x}_4\right) & \\ \quad \equiv -v_2 v_3^{-1-(p^2\omega-1)p^2}\widehat{t}_2^{p^5} & \bmod \left(v_2^{1+(p\omega-p^2-1)p}\right), \\ d\left(v_3^{-1-p^4\omega} v_4^{p^3\omega}\widehat{x}_0\right) & \\ \quad = v_3^{-1-p^4\omega} v_4^{p^3\omega}\left(v_2\widehat{t}_1^{p^2} - v_2^{p\omega}\widehat{t}_1\right), & \\ d\left(v_2^{1+p^3-\widehat{a}(4)} v_3^{-1-p^4\omega} v_4^{p^2}\widehat{x}_4\right) & \\ \quad \equiv v_2^{1+p^3} v_3^{-1-(p\omega+1)p^3} v_4^{p^2}\widehat{t}_2^{p^5} & \bmod \left(v_2^{1+(p\omega-1)p}\right), \\ d\left(-v_2^{1-p} v_3^{-1-p^3} v_4^{p^2}\widehat{x}_1\right) & \\ \quad \equiv -v_2 v_3^{-1-p^3} v_4^{p^2}\widehat{t}_1^{p^3} & \bmod \left(v_2^{1+(p\omega-1)p}\right), \\ d\left(v_2^{(p+1)p^2} v_3^{p^3\omega(1-p)-1-p^2}\widehat{v}_3^{(p-1)p^3}\widehat{x}_0\right) & \\ \quad \equiv v_2^{1+(p+1)p^2} v_3^{p^3\omega(1-p)-1-p^2}\widehat{v}_3^{(p-1)p^3}\widehat{t}_1^{p^2} & \bmod \left(v_2^{(2p+1)p^2}\right). \end{cases}$$

The sum of the right sides is

$$-v_3^{p\omega-1}\widehat{t}_1 + v_2^{p^2+1} v_3^{-1-p^4\omega}\widehat{t}_3^{p^4} + v_2^{(p+2)p^2+1} v_3^{-1-(p^2\omega+1)p^2}\widehat{v}_3^{(p-1)p^3}\widehat{t}_2^{p^4} \\ - v_2^{p\omega} v_3^{-1-p^4\omega} v_4^{p^3\omega}\widehat{t}_1.$$

Multiplying by $-v_3^{p^4\omega+1}$ gives the desired formula. The integer $e_2(p, m)$ is the minimum exponent of $v_2$ in these indeterminacies. □

Now we define $\widehat{w}_6$ by

$$\widehat{w}_6 = \begin{cases} v_3^{-1}\left(\widehat{v}_6 - v_2^{-p\widehat{a}(2)} v_5\widehat{x}_2^p + v_2^{-\widehat{a}(3)} v_3^{-\widehat{b}(3)} v_4\widehat{x}_3\right) & \text{for } m \geq 3, \\ \begin{pmatrix}\text{above}\\ \text{terms}\end{pmatrix} + v_2^{1-p^3} v_3^{-1-p^3\omega(p^3+1)}\widehat{x}_2^p Y^{p^2} - v_3^{-1-p\omega(p^3+1)}\widehat{x}_2^p Y & \text{for } m = 2. \end{cases}$$

**Lemma 5.4.** *In the module $v_3^{-1}(BP_*/I_2 \otimes \Gamma(m+1))$ we have*

$$d(\widehat{w}_6) \equiv \widehat{t}_3^{p^3} - v_3^{-1} v_5^{p\omega}\widehat{t}_1 - v_3^{-1} v_4^{p^2\omega}\widehat{t}_2 - v_3^{p^3\omega-1}\widehat{t}_3 \quad \bmod (v_2).$$

*Proof.* We find that

$$\left\{\begin{array}{rcl} d(\widehat{v}_6) & \equiv & v_5\widehat{t}_1^{p^5} + v_4\widehat{t}_2^{p^4} + v_3\widehat{t}_3^{p^3} - v_5^p\widehat{t}_1 - v_4^{p^2\omega}\widehat{t}_2 - v_3^{p^3\omega}\widehat{t}_3 \\ & & \qquad\qquad\qquad \mod (v_2)\,, \\ d(v_2^{-\widehat{a}(3)}v_3^{-\widehat{b}(3)}v_4\widehat{x}_3) & \equiv & -v_4\widehat{t}_2^{p^4} \qquad\qquad\qquad \mod \left(v_2^{p\omega-p^2-1}\right). \end{array}\right. \tag{5.5}$$

Notice that $\eta_R(v_5) = v_5$ for $m \geq 3$, and so we have

$$\begin{aligned} d\left(-v_2^{-p\widehat{a}(2)}v_5\widehat{x}_2^p\right) &= -v_2^{-p\widehat{a}(2)}\left(d(v_5)\widehat{x}_2^p + \eta(v_5)d(\widehat{x}_2^p)\right) \\ &= -v_2^{-p\widehat{a}(2)}v_5 d(\widehat{x}_2^p) \\ &\equiv -v_5\widehat{t}_1^{p^5} \qquad \mod \left(v_2^{(p\omega-1)p^3}\right). \end{aligned} \tag{5.6}$$

Multiplying the sum of (5.5) and (5.6) by $v_3^{-1}$, we obtain the desired formula for $m \geq 3$. For $m = 2$, we know that $\eta_R(v_5) = \eta_R(\widehat{v}_3) = v_5 + v_2\widehat{t}_1^{p^2} - v_2^{p^3}\widehat{t}_1$, and so we have

$$\begin{aligned} d\left(-v_2^{-p^3}v_5\widehat{x}_2^p\right) &\equiv -v_2^{-p^3}\left((v_2\widehat{t}_1^{p^2} - v_2^{p^3}\widehat{t}_1)\widehat{x}_2^p + (v_5 + v_2\widehat{t}_1^{p^2} - v_2^{p^3}\widehat{t}_1)(v_2^{p^3}\widehat{t}_1^{p^5})\right) \\ &\equiv -v_2^{-p^3}\left((v_2\widehat{t}_1^{p^2} - v_2^{p^3}\widehat{t}_1)\widehat{x}_2^p + v_2^{p^3}v_5\widehat{t}_1^{p^5}\right) \\ &\equiv -v_2^{1-p^3}\widehat{x}_2^p\widehat{t}_1^{p^2} + \widehat{x}_2^p\widehat{t}_1 - v_5\widehat{t}_1^{p^5} \qquad\qquad \mod (v_2)\,. \end{aligned} \tag{5.7}$$

We also find that

$$\left\{\begin{array}{rcl} d\left(v_2^{1-p^3}v_3^{-p^3\omega(p^3+1)}\widehat{x}_2^pY^{p^2}\right) & \equiv & v_2^{1-p^3}\widehat{x}_2^p\widehat{t}_1^{p^2}, \\ d\left(-v_3^{-p\omega(p^3+1)}\widehat{x}_2^pY\right) & \equiv & -\widehat{x}_2^p\widehat{t}_1 \qquad\qquad \mod (v_2)\,. \end{array}\right. \tag{5.8}$$

Multiplying the sum of (5.5), (5.7) and (5.8) by $v_3^{-1}$ gives the desired formula for $m = 2$. ∎

**Proposition 5.9.** *Let*

$$\begin{aligned} X \;=\; & \widehat{w}_6^{p^4} - v_2^{-\widehat{a}(2)}v_3^{-p^4}v_5^{p^5\omega}\widehat{x}_2 - v_2^{-\widehat{a}(3)}v_3^{-p^4+p^2}v_4^{p^6\omega}\widehat{x}_3 \\ & + v_2^{p\omega-p^2-1}v_3^{(\omega-p)p^3-p\omega(p^3+1)}v_4^{p^6\omega}Y. \end{aligned}$$

*Then for* $m \geq 3$,

$$d(X) \equiv \widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4}\widehat{t}_3^{p^4} \quad \mod \left(v_2^{p^4}\right).$$

*Proof.* We find the following congruences:

(5.10)
$$\begin{cases} d\left(\widehat{w}_6^{p^4}\right) \equiv \widehat{t}_3^{p^7} - v_3^{-p^4} v_5^{p^5\omega} \widehat{t}_1^{p^4} - v_3^{-p^4} v_4^{p^6\omega} \widehat{t}_2^{p^4} - v_3^{(p^3\omega-1)p^4} \widehat{t}_3^{p^4} \quad \bmod \left(v_2^{p^4}\right), \\ d\left(v_2^{-\widehat{a}(2)} v_3^{-p^4} v_5^{p^5\omega} \widehat{x}_2\right) \\ \qquad \equiv v_3^{-p^4} v_5^{p^5\omega} \widehat{t}_1^{p^4} \quad \bmod \left(v_2^{(p\omega-1)p^2}\right), \\ d\left(-v_2^{-\widehat{a}(3)} v_3^{-p^4+p^2} v_4^{p^6\omega} \widehat{x}_3\right) \\ \qquad \equiv -v_2^{-\widehat{a}(3)} v_3^{-p^4+p^2} v_4^{p^6\omega} \left(-v_2^{\widehat{a}(3)} v_3^{-p^2} \widehat{t}_2^{p^4} + v_2^{p^3+p\omega-1} v_3^{(p\omega-1)p^2} \widehat{t}_1\right) \\ \qquad \qquad \bmod \left(v_2^{(p^2\omega-p-1)p^2}\right), \\ d\left(v_2^{p\omega-p^2-1} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega} Y\right) \\ \qquad \equiv v_2^{p\omega-p^2-1} v_3^{(\omega-p)p^3} v_4^{p^6\omega} \widehat{t}_1 \quad \bmod \left(v_2^{p\omega}\right) \end{cases}$$

Summing these congruences gives the desired formula. ∎

In the case $m = 2$ we have the following analogous statment instead.

**Proposition 5.11.** *For $m = 2$, there is an element $\widetilde{X}$ such that*

$$d(\widetilde{X}) \equiv \widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4} \widehat{t}_3^{p^4} - v_2^{p^3} v_3^{p\omega(p^2-p^3-1-p^6)} v_4^{p^6\omega} \widehat{t}_3^{p^7} \quad \bmod \left(v_2^{2p^3-p^2-1}\right).$$

*Proof.* Instead of the last congruence of (5.10) we find that

$$\begin{aligned} & d\left(v_2^{p\omega-p^2-1} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega} Y\right) \\ & \qquad \equiv v_2^{p^3-p^2-1} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega} \left(v_3^{p\omega(p^3+1)} \widehat{t}_1 - v_2^{p^2+1} \widehat{t}_3^{p^4}\right) \\ & \qquad \qquad \bmod \left(v_2^{2p\omega-p^2-1}\right). \end{aligned}$$

This gives

$$d(X) \equiv \widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4} \widehat{t}_3^{p^4} - v_2^{p\omega} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega} \widehat{t}_3^{p^4} \quad \bmod \left(v_2^{2p^3-p^2-1}\right).$$

Then, define $\widetilde{X}$ by

$$\widetilde{X} = X - v_2^{p^3} v_3^{p\omega(p^2-p^3-1-p^6)} v_4^{p^6\omega} X.$$

This gives the desired formula. ∎

Define the element $M$ by

$$\begin{aligned} M = {} & \widehat{x}_5^p + v_2^{(p+1)p^5} v_3^{-p^5} W + v_2^{(p+1)p^5+p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-1)(p^3+1)} Y \\ & - v_2^{(p+1)p^5+2p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-2)(p^3+1)} v_4^{p^3\omega} Y. \end{aligned}$$

**Proposition 5.12.** *For $m \geq 2$, we have*

$$\begin{aligned}
d(M) \quad \equiv \quad & v_2^{p^4(p^2+p+1)} v_3^{-(p+1)p^4} \left(\widehat{t}_3^{p^6} - \widehat{x}_5^{p-1} d(\widehat{x}_5)\right) \\
& - v_2^{(p+1)p^5+p\omega} v_3^{-(p+1)p^4+p\omega(p^2-1)(p^3+1)} \left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)\right) \\
& + v_2^{(p+1)p^5+2p\omega} v_3^{-(p+1)p^4+p\omega(p^2-2)(p^3+1)} v_4^{p^3\omega} \left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)\right) \\
& - v_2^{(p+1)p^5+3p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-2)(p^3+1)} v_4^{2p^3\omega} \widehat{t}_1 \\
& \qquad \qquad \qquad \qquad \mod \left(v_2^{e(p,m)}\right).
\end{aligned}$$

*where*

$$e(p,m) = (p+1)p^5 + \min \left\{ \begin{array}{l} (p\omega - p^2 - 1)p^3, \\ e_1(p,m), \\ p\omega - p^2 - 1 + e_2(p,m) \end{array} \right\}.$$

*Proof.* We find that

$$\left\{ \begin{array}{l}
d\left(\widehat{x}_5^p\right) \quad \equiv \quad -v_2^{(p+1)p^5} v_3^{-p^5} \widehat{t}_2^{p^7} \qquad \qquad \mod \left(v_2^{(p^3+p\omega-1)p^3}\right), \\
d\left(v_2^{(p+1)p^5} v_3^{-p^5} W\right) \\
\qquad \equiv \quad v_2^{(p+1)p^5} v_3^{-p^5} \widehat{t}_2^{p^7} + v_2^{p^4(p^2+p+1)} v_3^{-(p+1)p^4} \left(\widehat{t}_3^{p^6} - \widehat{v}_3^{(p-1)p^5} d(\widehat{x}_5)\right) \\
\qquad \qquad -v_2^{(p+1)p^5+p\omega-p^2-1} v_3^{(p^2\omega-p-1)p^4+p^3\omega} \widehat{t}_1 \quad \mod \left(v_2^{(p+1)p^5+e_1(p,m)}\right), \\
d\left(v_2^{(p+1)p^5+p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-1)(p^3+1)} Y\right) \\
\qquad \equiv \quad v_2^{(p+1)p^5+p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-2)(p^3+1)} \widehat{t}_1 \\
\qquad \qquad -v_2^{(p+1)p^5+p\omega} v_3^{-(p+1)p^4+p\omega(p^2-1)(p^3+1)} \left(\widehat{t}_3^{p^4} - \widehat{v}_3^{(p-1)p^3} d(\widehat{x}_3)\right) \\
\qquad \qquad +v_2^{(p+1)p^5+2p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-1)(p^3+1)} v_4^{p^3\omega} \widehat{t}_1 \\
\qquad \qquad \qquad \qquad \mod \left(v_2^{(p+1)p^5+p\omega-p^2-1+e_2(p,m)}\right), \\
d\left(-v_2^{(p+1)p^5+2p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-2)(p^3+1)} v_4^{p^3\omega} Y\right) \\
\qquad \equiv \quad -v_2^{(p+1)p^5+2p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-1)(p^3+1)} v_4^{p^3\omega} \widehat{t}_1 \\
\qquad \qquad +v_2^{(p+1)p^5+2p\omega} v_3^{-(p+1)p^4+p\omega(p^2-2)(p^3+1)} v_4^{p^3\omega} \left(\widehat{t}_3^{p^4} - \widehat{v}_3^{(p-1)p^3} d(\widehat{x}_3)\right) \\
\qquad \qquad -v_2^{(p+1)p^5+3p\omega-p^2-1} v_3^{-(p+1)p^4+p\omega(p^2-2(p^3+1)} v_4^{2p^3\omega} \widehat{t}_1 \\
\qquad \qquad \qquad \qquad \mod \left(v_2^{(p+1)p^5+2p\omega-p^2-1+e_2(p,m)}\right).
\end{array} \right.$$

Summing these congruences gives the desired formula. □

## 6. $d(\widehat{x}_k)$ for $k \geq 6$

Define integers $\widehat{a}(k)$ and $\widehat{b}(k)$ for $k \geq 6$ (they were defined for $0 \leq k \leq 5$ in (4.1)) by

(6.1)

$$\widehat{a}(k) = \begin{cases} p^{k-2}(p^2+p+1) & \text{for } m \geq 3 \text{ and } 6 \leq k \leq 8, \\ p^{k-3}(p^3+p^2+1) & \text{for } m = 2 \text{ and } 6 \leq k \leq 8, \\ p^3\widehat{a}(6) + \widehat{a}(5) & \text{for } m \geq 5 \text{ and } k = 9, \\ p^3\widehat{a}(6) + (p-1)p^4 + \widehat{a}(3) & \text{for } m = 4 \text{ and } k = 9, \\ p^3\widehat{a}(6) + \widehat{a}(3) & \text{for } 2 \leq m \leq 3 \text{ and } k = 9, \\ p^{k-9}(\widehat{a}(9) - \widehat{a}(5)) + \widehat{a}(k-4) & \text{for } m \geq 5 \text{ and } k \geq 10, \\ p^{k-9}(\widehat{a}(9) - \widehat{a}(3)) + \widehat{a}(k-6) & \text{for } 2 \leq m \leq 4 \text{ and } k \geq 10; \end{cases}$$

$$\widehat{b}(k) = \begin{cases} -p^{k-2}(p+1) & \text{for } m \geq 3 \text{ and } 6 \leq k \leq 8, \\ p^{k-3}(p^5-p^3-p-1) & \text{for } m = 2 \text{ and } 6 \leq k \leq 8, \\ p^3\widehat{b}(6) + \widehat{b}(5) + (p^3\omega - 1)p^6 & \text{for } m \geq 5 \text{ and } k = 9, \\ p^3\widehat{b}(6) + p^5(p^8+p^5-p^3+p^2-p-1) + \widehat{b}(3) & \\ & \text{for } m = 4 \text{ and } k = 9, \\ p^3\widehat{b}(6) + p^4(p^5-1)(p^3+1) + \widehat{b}(3) & \text{for } m = 3 \text{ and } k = 9, \\ p^3\widehat{b}(6) + p^4(p^5-1) + \widehat{b}(3) & \text{for } m = 2 \text{ and } k = 9, \\ p^{k-9}(\widehat{b}(9) - \widehat{b}(5)) + \widehat{b}(k-4) & \text{for } m \geq 5 \text{ and } k \geq 10, \\ p^{k-9}(\widehat{b}(9) - \widehat{b}(3)) + \widehat{b}(k-6) & \text{for } 2 \leq m \leq 4 \text{ and } k \geq 10. \end{cases}$$

According to (4.1) and (6.1), the definitions of $\widehat{a}(k)$ and $\widehat{b}(k)$ are summarized in the following tables.

Definition of $\widehat{a}(k)$

| $k$ | $m=2$ | $m=3$ | $m=4$ | $m \geq 5$ |
|---|---|---|---|---|
| $0 \leq k \leq 2$ | $p^k$ | | | |
| $3 \leq k \leq 5$ | $p^k + p^{k-1}$ | | | |
| $6 \leq k \leq 8$ | $p^k + p^{k-1} + p^{k-3}$ | $p^k + p^{k-1} + p^{k-2}$ | | |
| $k \geq 9$ | $p^k + p^{k-1} + p^{k-3}$ $+\widehat{a}(k-6)$ | $p^k + p^{k-1} + p^{k-2}$ $+\widehat{a}(k-6)$ | $p^k + p^{k-1} + p^{k-2}$ $+p^{k-4} - p^{k-5}$ $+\widehat{a}(k-6)$ | $p^k + p^{k-1} + p^{k-2}$ $+\widehat{a}(k-4)$ |

Definition of $\widehat{b}(k)$

| $k$ | $m=2$ | $m=3$ | $m=4$ | $m \geq 5$ |
|---|---|---|---|---|
| $0 \leq k \leq 2$ | $0$ | | | |
| $3 \leq k \leq 5$ | $-p^{k-1}$ | | | |
| $6 \leq k \leq 8$ | $p^{k-3}(p^5-p^3-p-1)$ | $-p^{k-2}(p+1)$ | | |
| $k \geq 9$ | $p^{k-5}(p^7-p^3-p^2-1)$ $+\widehat{b}(k-6)$ | $p^{k-5}(p^8+p^5$ $-p^4-2p^3-1)$ $+\widehat{b}(k-6)$ | $p^{k-4}(p^8+p^5$ $-2p^3-p-1)$ $+\widehat{b}(k-6)$ | $-p^{k-2}(p+1)$ $+(p^3\omega-1)p^{k-3}$ $+\widehat{b}(k-4)$ |

Define elements $\widehat{y}_i$ for $1 \le i \le 4$ by

$$\begin{cases} \widehat{y}_1 = -v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X^{p^2} - v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6}\widehat{x}_6, \\ \widehat{y}_2 = v_2^{p^3\widehat{a}(6)-p^4+p\omega} v_3^{p^3\widehat{b}(6)+p\omega(p^2-1)(p^3+1)+(p^3\omega-1)(p^6-p^4)} \left(X - v_2^{-\widehat{a}(7)} v_3^{-\widehat{b}(7)}\widehat{x}_7\right), \\ \widehat{y}_3 = v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X, \\ \widehat{y}_4 = v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X - v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^5-1)p^4}\widehat{x}_6. \end{cases}$$

For $k \ge 6$, define elements $\widehat{x}_k \in v_3^{-1}BP_*$ for $m \ge 5$ by

$$\widehat{x}_k = \begin{cases} M^{p^{k-6}} & \text{for all } m \text{ and } 6 \le k \le 8, \\ \widehat{x}_8^p + \widehat{y}_1 & \text{for } 4 \le m \le 5 \text{ and } k = 9 \\ & \text{ unless } (p, m) = (2, 5), \\ \widehat{x}_8^p + \widehat{y}_1 + \widehat{y}_2 & \text{for } (p, m, k) = (2, 5, 9) \\ & \text{ or } m = 4 \text{ and } k = 9, \\ \widehat{x}_8^p + \widehat{y}_1 + \widehat{y}_3 & \text{for } m = 3 \text{ and } k = 9, \\ \widehat{x}_8^p + \widehat{y}_4 & \text{for } m = 2 \text{ and } k = 9, \\ \widehat{x}_{k-1}^p + v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1}\left(\widehat{x}_{k-4} - \widehat{x}_{k-5}^p\right) & \\ & \text{for } m \ge 5 \text{ and } k \ge 10, \\ \widehat{x}_{k-1}^p - v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1}\left(\widehat{x}_{6-4} - \widehat{x}_{7-5}^p\right) & \\ & \text{for } 2 \le m \le 4 \text{ and } k \ge 10. \end{cases} \tag{6.2}$$

Then we have

**Lemma 6.3.** *Assume that $p$ and $m$ satisfy (5.1). Modulo $\left(v_2^{1+\widehat{a}(k)}\right)$, the differential on $\widehat{x}_k$ $(k \ge 6)$ is*

$$v_2^{\widehat{a}(k)} v_3^{\widehat{b}(k)} \begin{cases} \widehat{t}_3^{p^k} & \text{for } m \ge 4 \text{ and } 6 \le k \le 8, \\ \widehat{t}_3^{p^k} - v_3^{p^{k-2}(p^2-1)(p^3+1)}\widehat{t}_3^{p^{k-2}} & \text{for } m = 3 \text{ and } 6 \le k \le 8, \\ -\widehat{t}_3^{p^{k-2}} & \text{for } m = 2 \text{ and } 6 \le k \le 8, \\ v_2^{-\widehat{a}(k-4)} v_3^{-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-4}) & \text{for } m \ge 5 \text{ and } k \ge 9, \\ -v_2^{-\widehat{a}(k-6)} v_3^{-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{x}_{k-6}) & \text{for } 2 \le m \le 4 \text{ and } k \ge 9. \end{cases}$$

□

According to Lemma 4.3 and Lemma 6.3, the elements of $d(\widehat{x}_k)$ are summarized in the following table:

| $k = 2$ | $m = 2$ | $m = 3$ | $m = 4$ | $m \ge 5$ |
|---|---|---|---|---|
| $0 \le k \le 2$ | $\widehat{t}_1^{p^{k+2}}$ | | | |
| $3 \le k \le 5$ | $-\widehat{t}_2^{p^{k+1}}$ | | | |
| $6 \le k \le 8$ | $-\widehat{t}_3^{p^{k-2}}$ | $\widehat{t}_3^{p^k} - v_3^{p^{k-2}(p^2-1)(p^3+1)}\widehat{t}_3^{p^{k-2}}$ | $\widehat{t}_3^{p^k}$ | |
| $k \ge 9$ | $-v_2^{-\widehat{a}(k-6)} v_3^{-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{x}_{k-6})$ | | | $v_2^{-\widehat{a}(k-4)} v_3^{-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-4})$ |

We will prove this lemma in Sections 7, 8, 9 and 10. Then our main result is

**Theorem 6.4.** *As a $\widehat{k}(2)_*$-module, $\mathrm{Ext}_{\Gamma(m+1)}(M_2^1)$ is the direct sum of*

1. *the cyclic submodules freely generated by $\widehat{x}_k^s / v_2^{\widehat{a}(k)}$ for $k \ge 0$ and $p \nmid s > 0$, where $\widehat{x}_k$ and $\widehat{a}(k)$ are the elements defined in (4.1), (4.2), (6.1) and (6.2); and*

2. $\widehat{K}(2)_*/\widehat{k}(2)_*$, *generated by* $1/v_2^j$ *for* $j \geq 1$.

## 7. Proof of Lemma 6.3 for $m \geq 5$

Notice that the exponent $e(p, m)$ in Proposition 5.12 is

$$(p+1)p^5 + \min\left\{\begin{array}{l}(p\omega - p^2 - 1)p^3,\\ (2p+1)p^4 - p^2,\\ p\omega + 2p^3 - 1\end{array}\right\} = (p+1)p^5 + (2p+1)p^4 - p^2,$$

which is always larger than

$$\widehat{a}(6) + \widehat{a}(5) + p^3 + p + 2 = p^6 + 2p^5 + 2p^4 + p^3 + p + 2$$

for all primes $p$. By Proposition 5.12 we have

$$d(\widehat{x}_6) \equiv \begin{cases} v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \left(\widehat{t}_3^{64} - \widehat{x}_5 d(\widehat{x}_5)\right) - v_2^{\widehat{a}(6)+48} v_3^{\widehat{b}(6)+1728} \widehat{t}_3^{16} & \\ & \text{if } (p, m) = (2, 5), \\ v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \left(\widehat{t}_3^{p^6} - \widehat{x}_5^{p-1} d(\widehat{x}_5)\right) & \text{otherwise,} \end{cases}$$

mod $\left(v_2^{\widehat{a}(6)+\widehat{a}(5)+p^3+p+2}\right)$. In particular, we have

$$d(\widehat{x}_6) \equiv v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \widehat{t}_3^{p^6} \qquad \bmod \left(v_2^{\widehat{a}(6)+\widehat{a}(5)}\right)$$

in both cases. It follows that

$$d(\widehat{x}_8^p) \equiv v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \widehat{t}_3^{p^9} \qquad \bmod \left(v_2^{p^3(\widehat{a}(6)+\widehat{a}(5))}\right). \tag{7.1}$$

Assume that $(p, m) \neq (2, 5)$. For $d(\widehat{y}_1)$ we find that

$$\begin{cases} d\left(-v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X^{p^2}\right) \\ \qquad \equiv \quad -v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \left(\widehat{t}_3^{p^9} - v_3^{(p^3\omega-1)p^6} \widehat{t}_3^{p^6}\right) \quad \bmod \left(v_2^{p^3\widehat{a}(6)+p^6}\right), \\ d\left(-v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6} \widehat{x}_6\right) \\ \qquad \equiv \quad -v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)+(p^3\omega-1)p^6} \left(\widehat{t}_3^{p^6} - \widehat{x}_5^{p-1} d(\widehat{x}_5)\right) \\ \qquad\qquad \bmod \left(v_2^{p^3\widehat{a}(6)+\widehat{a}(5)+p^3+p+2}\right). \end{cases} \tag{7.2}$$

(Notice that the second formula fails if $(p, m) = (2, 5)$.) This gives

$$d(\widehat{y}_1) \equiv v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \left(v_3^{(p^3\omega-1)p^6} \widehat{x}_5^{p-1} d(\widehat{x}_5) - \widehat{t}_3^{p^9}\right) \tag{7.3}$$

modulo $(v_2^{p^3\widehat{a}(6)+\widehat{a}(5)+p^3+p+2})$. Summing (7.1) and (7.3), we obtain

$$d(\widehat{x}_9) \equiv v_2^{\widehat{a}(9)-\widehat{a}(5)} v_3^{\widehat{b}(9)-\widehat{b}(5)} \widehat{x}_5^{p-1} d(\widehat{x}_5) \qquad \bmod \left(v_2^{\widehat{a}(9)+p^3+p+2}\right)$$

unless $(p, m) = (2, 5)$. We will see that a similar congruence holds even for $(p, m) = (2, 5)$ after an appropriate change of $\widehat{x}_9$.

Define integers $n(k)$ by

$$n(k) = \begin{cases} p^3+p+2 & \text{for } k \equiv 1 \ (4), \\ (p+2)p & \text{for } k \equiv 2 \ (4), \\ (p+2)p^2 & \text{for } k \equiv 3 \ (4), \\ (p+2)p^3 & \text{for } k \equiv 0 \ (4). \end{cases}$$

Instead of (6.1), we may define integers $\widehat{a}(k)$ for $k \geq 9$ inductively on $k$ by

$$\widehat{a}(k) = \begin{cases} p\widehat{a}(k-1)+p^5+p^4 & \text{for } k \equiv 1 \bmod (4), \\ p\widehat{a}(k-1)+p^4 & \text{for } k \equiv 2 \bmod (4), \\ p\widehat{a}(k-1) & \text{for } k \equiv 0 \text{ and } 3 \bmod (4). \end{cases}$$

We will compute $d(\widehat{x}_k)$ modulo $(v_2^{\widehat{a}(k)+n(k)})$ inductively on $k$. Assume that the congruence

$$d(\widehat{x}_{k-1}) \equiv v_2^{p^{k-10}(\widehat{a}(9)-\widehat{a}(5))} v_3^{p^{k-10}(\widehat{b}(9)-\widehat{b}(5))} \widehat{x}_{k-5}^{p-1} d(\widehat{x}_{k-5}) \tag{7.4}$$

holds modulo $(v_2^{\widehat{a}(k-1)+n(k-1)})$. For $10 \leq k \leq 14$ it follows by direct calculations. Moreover, this gives $d(\widehat{x}_k)$ whenever $11 \leq k \equiv 0$ or $3 \bmod (4)$, since $\widehat{x}_k = \widehat{x}_{k-1}^p$. In other cases we denote $\widehat{x}_k - \widehat{x}_{k-1}^p$ by $\widehat{z}_k$. Notice that $\widehat{z}_k$ is related to $\widehat{z}_{k-4}$ by

$$\widehat{z}_k = v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} \widehat{z}_{k-4}.$$

Then $d(\widehat{z}_k)$ is

$$\begin{aligned} d(\widehat{z}_k) &= v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} d\left(\widehat{x}_{k-4}^{p-1}\widehat{z}_{k-4}\right) \\ &= v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \left(d\left(\widehat{x}_{k-4}^{p-1}\right)\widehat{z}_{k-4} + \eta_R\left(\widehat{x}_{k-4}^{p-1}\right) d\left(\widehat{z}_{k-4}\right)\right). \end{aligned} \tag{7.5}$$

Notice that $\widehat{z}_{k-4}$ is divisible by $v_2^{\widehat{a}(k-4)-\widehat{a}(k-8)}$, so that

$$\begin{aligned} & v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \cdot d\left(\widehat{x}_{k-4}^{p-1}\right)\widehat{z}_{k-4} \\ &= v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \cdot d\left(\widehat{x}_{k-4}^{p-1}\right) v_2^{\widehat{a}(k-4)-\widehat{a}(k-8)} z \\ &= v_2^{\widehat{a}(k)-\widehat{a}(k-8)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} z \cdot d\left(\widehat{x}_{k-4}^{p-1}\right) \end{aligned}$$

for some $z$. The facts that $d\left(\widehat{x}_{k-4}^{p-1}\right)$ is trivial in mod $\left(v_2^{\widehat{a}(k-4)}\right)$ and that

$$\widehat{a}(k)+\widehat{a}(k-4)-\widehat{a}(k-8) \geq \widehat{a}(k)+n(k)$$

imply that we can ignore the first term of (7.5) whenever we work mod $\left(v_2^{\widehat{a}(k)+n(k)}\right)$. The similar statement is satisfied for the second term. Thus we have

$$d\left(\widehat{z}_k\right) \equiv v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{z}_{k-4}) \qquad \bmod \left(v_2^{\widehat{a}(k)+n(k)}\right). \tag{7.6}$$

On the other hand, by assumption (7.4) we have

$$\begin{aligned} d(\widehat{x}_{k-1}^p) &\equiv v_2^{p^{k-9}(\widehat{a}(9)-\widehat{a}(5))} v_3^{p^{k-9}(\widehat{b}(9)-\widehat{b}(5))} \widehat{x}_{k-5}^{(p-1)p} d(\widehat{x}_{k-5}^p) \\ &\equiv v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-5}^p) \qquad \bmod \left(v_2^{\widehat{a}(k)+n(k)}\right). \end{aligned} \tag{7.7}$$

Summing (7.6) and (7.7), we obtain

$$d(\widehat{x}_k) \equiv v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-4}) \qquad \mathrm{mod}\left(v_2^{\widehat{a}(k)+n(k)}\right),$$

as desired. □

Now we consider the $(p, m) = (2, 5)$ case. We have the following congruence instead of the second one in (7.3):

$$\begin{aligned} &d\left(-v_2^{7\widehat{a}(6)} v_3^{7\widehat{b}(6)+16320} \widehat{x}_6\right) \\ &\quad \equiv \quad v_2^{8\widehat{a}(6)} v_3^{8\widehat{b}(6)+16320} \left\{ v_2^{48} v_3^{1728} \widehat{t}_3^{16} - \left(\widehat{t}_3^{64} - \widehat{x}_5 d(\widehat{x}_5)\right)\right\} \quad \mathrm{mod}\left(v_2^{\widehat{a}(9)+12}\right). \end{aligned} \tag{7.8}$$

In respect to the computation of $d(\widehat{y}_2)$ we see that the differential on $X - v_2^{-\widehat{a}(7)} v_3^{-\widehat{b}(7)} \widehat{x}_7$ is expressed as

$$d\left(X - v_2^{-\widehat{a}(7)} v_3^{-\widehat{b}(7)} \widehat{x}_7\right) \equiv -v_3^{4080} \widehat{t}_3^{16} \qquad \mathrm{mod}\left(v_2^{16}\right).$$

This gives

$$d(\widehat{y}_2) \equiv -v_2^{8\widehat{a}(6)+48} v_3^{8\widehat{b}(6)+18048} \widehat{t}_3^{16} \tag{7.9}$$

mod $\left(v_2^{8\widehat{a}(6)+64}\right)$. Summing (7.8) and (7.9), we have the same formula as the second of (7.2). □

## 8. Proof of Lemma 6.3 for $m = 4$

Notice that the exponent $e(p, m)$ in Proposition 5.12 is

$$(p+1)p^5 + \min \left\{ \begin{array}{l} (p^5 - p^2 - 1)p^3, \\ (2p+1)p^4 - p^2, \\ p^5 + 2p^3 - 1 \end{array} \right\} = p^6 + 2p^5 + 2p^3 - 1,$$

which is always greater than or equal to

$$(p+1)p^5 + p\omega + \widehat{a}(3) + p + 1 = p^6 + 2p^5 + p^3 + p^2 + p + 1$$

for all primes $p$. By Proposition 5.12 we have

$$d(\widehat{x}_6) \equiv v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \widehat{t}_3^{p^6} - v_2^{\widehat{a}(6)+(p-1)p^4} v_3^{\widehat{b}(6)+p^5(p^2-1)(p^3+1)} \left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)\right)$$

mod $\left(v_2^{\widehat{a}(6)+(p-1)p^4+\widehat{a}(3)+p+1}\right)$. In particular, we have

$$d(\widehat{x}_6) \equiv v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \widehat{t}_3^{p^6} \quad \mathrm{mod}\left(v_2^{\widehat{a}(6)+(p-1)p^4}\right),$$

and it follows that

$$d(\widehat{x}_8^p) \equiv v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \widehat{t}_3^{p^9} \qquad \mathrm{mod}\left(v_2^{p^3\widehat{a}(6)+(p-1)p^7}\right). \tag{8.1}$$

For $d(\widehat{y}_1)$ we find that

$$\begin{cases} d\left(-v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}X^{p^2}\right) \\ \quad\equiv \quad -v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}\left(\widehat{t}_3^{p^9}-v_3^{(p^7-1)p^6}\widehat{t}_3^{p^6}\right) \quad \bmod \left(v_2^{p^3\widehat{a}(6)+p^6}\right), \\ d\left(-v_2^{(p^3-1)\widehat{a}(6)}v_3^{(p^3-1)\widehat{b}(6)+(p^7-1)p^6}\widehat{x}_6\right) \\ \quad\equiv \quad v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)+(p^7-1)p^6} \\ \qquad\qquad \left(v_2^{(p-1)p^4}v_3^{p^5(p^2-1)(p^3+1)}\left(\widehat{t}_3^{p^4}-\widehat{x}_3^{p-1}d(\widehat{x}_3)\right)-\widehat{t}_3^{p^6}\right) \\ \qquad\qquad\qquad \bmod \left(v_2^{p^3\widehat{a}(6)+(p-1)p^4+\widehat{a}(3)+p+1}\right). \end{cases}$$

This gives

(8.2)
$$d(\widehat{y}_1) \equiv v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}\left\{v_2^{(p-1)p^4}v_3^{p^5(p^8+p^5-p^3+p^2-p-1)}\left(\widehat{t}_3^{p^4}-\widehat{x}_3^{p-1}d(\widehat{x}_3)\right)-\widehat{t}_3^{p^9}\right\}$$

mod $\left(v_2^{p^3\widehat{a}(6)+(p-1)p^4+\widehat{a}(3)+p+1}\right)$. Moreover, in the same way as (7.9) we have

$$d(\widehat{y}_2) \equiv -v_2^{p^3\widehat{a}(6)+(p-1)p^4}v_3^{p^3\widehat{b}(6)+p^5(p^8+p^5-p^3+p^2-p-1)}\widehat{t}_3^{p^4} \tag{8.3}$$

mod $\left(v_2^{p^3\widehat{a}(6)+p^5}\right)$. Summing (8.1), (8.2) and (8.3), we obtain

$$d(\widehat{x}_9) \equiv -v_2^{\widehat{a}(9)-\widehat{a}(3)}v_3^{\widehat{b}(9)-\widehat{b}(3)}\widehat{x}_3^{p-1}d(\widehat{x}_3) \qquad \bmod \left(v_2^{\widehat{a}(9)+p+1}\right).$$

We will compute $d(\widehat{x}_k)$ modulo $\left(v_2^{\widehat{a}(k)+n(k)}\right)$ inductively for $k \geq 10$. Define integers $n(k)$ by

$$n(k) = \begin{cases} p+1 & \text{for } k \equiv 3\ (6), \\ (p+1)p^2 & \text{for } k \equiv 4\ (6), \\ (p+1)p^3 & \text{for } k \equiv 5\ (6), \\ p^2-p+2 & \text{for } k \equiv 0\ (6), \\ (p^2-p+2)p & \text{for } k \equiv 1\ (6), \\ (p^2-p+2)p^2 & \text{for } k \equiv 2\ (6). \end{cases}$$

Instead of (6.1), we may define integers $\widehat{a}(k)$ for $k \geq 9$ inductively on $k$ by

$$\widehat{a}(k) = \begin{cases} p\widehat{a}(k-1)+p^5-p^4+p^3+p^2 & \text{for } k \equiv 3 \bmod (6), \\ p\widehat{a}(k-1)+p^4 & \text{for } k \equiv 0 \bmod (6), \\ p\widehat{a}(k-1) & \text{for } k \not\equiv 0 \bmod (3). \end{cases}$$

Assume that the congruences

$$d(\widehat{x}_{k-1}) \equiv -v_2^{p^{k-10}(\widehat{a}(9)-\widehat{a}(3))}v_3^{p^{k-10}(\widehat{b}(9)-\widehat{b}(3))}\widehat{x}_{k-7}^{p-1}d(\widehat{x}_{k-7}) \tag{8.4}$$

hold modulo $\left(v_2^{\widehat{a}(k-1)+n(k-1)}\right)$. For $10 \leq k \leq 16$ the case (8.4) follows by direct calculations. Moreover, this gives $d(\widehat{x}_k)$ whenever $10 \leq k \not\equiv 0$ modulo (3), since

$\widehat{x}_k = \widehat{x}_{k-1}^p$. In other cases we denote $\widehat{x}_k - \widehat{x}_{k-1}^p$ by $\widehat{z}_k$. Notice that $\widehat{z}_k$ is related to $\widehat{z}_{k-6}$ by

$$\widehat{z}_k = -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} \widehat{z}_{k-6}.$$

Then $d(\widehat{z}_k)$ is expressed as

$$\begin{aligned} d\left(\widehat{z}_k\right) &= -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} d\left(\widehat{x}_{k-6}^{p-1} \widehat{z}_{k-6}\right) \\ &= -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \left\{ d\left(\widehat{x}_{k-6}^{p-1}\right) \widehat{z}_{k-6} + \eta_R\left(\widehat{x}_{k-6}^{p-1}\right) d\left(\widehat{z}_{k-6}\right)\right\}. \end{aligned}$$

Notice that $\widehat{z}_{k-6}$ is divisible by $v_2^{\widehat{a}(k-6)-\widehat{a}(k-12)}$, so that we can ignore $d\left(\widehat{x}_{k-6}^{p-1}\right)$ as in the case $m \geq 5$. Thus we have

$$d\left(\widehat{z}_k\right) \equiv -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{z}_{k-6}) \qquad \operatorname{mod}\left(v_2^{\widehat{a}(k)+n(k)}\right). \tag{8.5}$$

On the other hand, by assumption (8.4) we have

$$\begin{aligned} d(\widehat{x}_{k-1}^p) &\equiv -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-7}^{(p-1)p} d(\widehat{x}_{k-7}^p) \\ &\equiv -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{x}_{k-7}^p) \qquad \operatorname{mod}\left(v_2^{\widehat{a}(k)+n(k)}\right). \end{aligned} \tag{8.6}$$

Summing (8.5) and (8.6), we obtain

$$d(\widehat{x}_k) \equiv -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{x}_{k-6}) \qquad \operatorname{mod}\left(v_2^{\widehat{a}(k)+n(k)}\right),$$

as desired. □

## 9. Proof of Lemma 6.3 for $m = 3$

Notice that the exponent $e(p, m)$ in Proposition 5.12 is

$$(p+1)p^5 + \min\left\{\begin{array}{l} (p^4-p^2-1)p^3, \\ e_1(p,3), \\ p^4+2p^3-1 \end{array}\right\} = (p+1)p^5 + p^4 + 2p^3 - 1,$$

which is always greater than or equal to

$$(p+1)p^5 + p\omega + \widehat{a}(3) + p + 1 = p^6 + p^5 + p^4 + p^3 + p^2 + p + 1$$

for all primes $p$. By Proposition 5.12 we have

$$d(\widehat{x}_6) \equiv v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \widehat{t}_3^{p^6} - v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)+p^4(p^2-1)(p^3+1)} \left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)\right)$$

mod $\left(v_2^{\widehat{a}(6)+\widehat{a}(3)+p+1}\right)$. In particular, we have

$$d(\widehat{x}_6) \equiv v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \left(\widehat{t}_3^{p^6} - v_3^{p^4(p^2-1)(p^3+1)} \widehat{t}_3^{p^4}\right) \qquad \operatorname{mod}\left(v_2^{\widehat{a}(6)+\widehat{a}(3)}\right),$$

and it follows that

$$d(\widehat{x}_8^p) \equiv v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \left(\widehat{t}_3^{p^9} - v_3^{p^7(p^2-1)(p^3+1)} \widehat{t}_3^{p^7}\right) \tag{9.1}$$

mod $\left(v_2^{p^3(\widehat{a}(6)+\widehat{a}(3))}\right)$. For $d(\widehat{y}_1)$ we find that

$$\begin{cases} d\left(-v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}X^{p^2}\right) \\ \qquad \equiv \; -v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}\left(\widehat{t}_3^{p^9} - v_3^{(p^6-1)p^6}\widehat{t}_3^{p^6}\right) \quad \mathrm{mod}\left(v_2^{p^3\widehat{a}(6)+p^6}\right), \\ d\left(-v_2^{(p^3-1)\widehat{a}(6)}v_3^{(p^3-1)\widehat{b}(6)+(p^6-1)p^6}\widehat{x}_6\right) \\ \qquad \equiv \; v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)+(p^6-1)p^6}\left\{v_3^{p^4(p^2-1)(p^3+1)}\left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1}d(\widehat{x}_3)\right) - \widehat{t}_3^{p^6}\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathrm{mod}\left(v_2^{p^3\widehat{a}(6)+\widehat{a}(3)+p+1}\right). \end{cases}$$

This gives

$$d(\widehat{y}_1) \equiv v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}\left\{v_3^{p^4(p^5-1)(p^3+1)}\left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1}d(\widehat{x}_3)\right) - \widehat{t}_3^{p^9}\right\} \tag{9.2}$$

mod $\left(v_2^{p^3\widehat{a}(6)+\widehat{a}(3)+p+1}\right)$. Moreover, we see that

$$d(\widehat{y}_3) \equiv v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)+p^7(p^2-1)(p^3+1)}\left(\widehat{t}_3^{p^7} - v_3^{(p^6-1)p^4}\widehat{t}_3^{p^4}\right) \tag{9.3}$$

mod $\left(v_2^{p^3\widehat{a}(6)+p^4}\right)$. Summing (9.1), (9.2) and (9.3), we obtain

$$d(\widehat{x}_9) \equiv -v_2^{\widehat{a}(9)-\widehat{a}(3)}v_3^{\widehat{b}(9)-\widehat{b}(3)}\widehat{x}_3^{p-1}d(\widehat{x}_3) \qquad \mathrm{mod}\left(v_2^{\widehat{a}(9)+p+1}\right).$$

We will compute $d(\widehat{x}_k)$ modulo $\left(v_2^{\widehat{a}(k)+n(k)}\right)$ inductively for $k \geq 10$. Define integers $n(k)$ by

$$n(k) = \begin{cases} p+1 & \text{for } k \equiv 3 \ (6), \\ (p+1)p & \text{for } k \equiv 4 \ (6), \\ (p+1)p^2 & \text{for } k \equiv 5 \ (6), \\ p^3 & \text{for } k \equiv 0 \ (6), \\ p^4 & \text{for } k \equiv 1 \ (6), \\ p^5 & \text{for } k \equiv 2 \ (6). \end{cases}$$

Instead of (6.1), we may define integers $\widehat{a}(k)$ for $k \geq 9$ inductively on $k$ by

$$\widehat{a}(k) = \begin{cases} p\widehat{a}(k-1) + p^3 + p^2 & \text{for } k \equiv 3 \bmod (6), \\ p\widehat{a}(k-1) + p^4 & \text{for } k \equiv 0 \bmod (6), \\ p\widehat{a}(k-1) & \text{for } k \not\equiv 0 \bmod (3). \end{cases}$$

We can prove this proposition in the same fashion as in the case $m = 4$. □

## 10. Proof of Lemma 6.3 for $m = 2$

Notice that the exponent $e(p, m)$ in Proposition 5.12 is

$$(p+1)p^5 + \min\left\{\begin{array}{l} (p^3-p^2-1)p^3, \\ (p^3-1)p^2, \\ p^3-p^2-1+e_2(p,2) \end{array}\right\} = p^6+p^5+p^3-p^2-1+e_2(p,2),$$

which is larger than

$$(p+1)p^5 + p\omega + \widehat{a}(3) + 2 = p^6 + p^5 + 2p^3 + p^2 + 2$$

only if $p > 2$ (and we may consider only these cases because of condition (5.1)). By Proposition 5.12 we have

$$d(\widehat{x}_6) \equiv -v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)\right)$$

mod $\left(v_2^{\widehat{a}(6)+\widehat{a}(3)+2}\right)$. In particular, we have

$$d(\widehat{x}_6) \equiv -v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \widehat{t}_3^{p^4} \qquad \bmod \left(v_2^{\widehat{a}(6)+\widehat{a}(3)}\right),$$

and it follows that

$$d(\widehat{x}_8^p) \equiv -v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \widehat{t}_3^{p^7} \qquad \bmod \left(v_2^{p^3(\widehat{a}(6)+\widehat{a}(3))}\right). \tag{10.1}$$

For $d(\widehat{y}_4)$ we find that

$$\begin{cases} d(v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X) & \equiv & v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \left(\widehat{t}_3^{p^7} - v_3^{(p^5-1)p^4} \widehat{t}_3^{p^4}\right) \\ & & \qquad \bmod \left(v_2^{p^3\widehat{a}(6)+p^4}\right), \\ d(-v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^5-1)p^4} \widehat{x}_6) & & \\ & \equiv & v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)+(p^5-1)p^4} \left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)\right) \\ & & \qquad \bmod \left(v_2^{p^3\widehat{a}(6)+\widehat{a}(3)+2}\right). \end{cases}$$

This gives

$$d(\widehat{y}_4) \equiv v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \left(\widehat{t}_3^{p^7} - v_3^{(p^5-1)p^4} \widehat{x}_3^{p-1} d(\widehat{x}_3)\right) \tag{10.2}$$

mod $\left(v_2^{p^3\widehat{a}(6)+\widehat{a}(3)+2}\right)$. Summing (10.1) and (10.2), we obtain

$$d(\widehat{x}_9) \equiv -v_2^{\widehat{a}(9)-\widehat{a}(3)} v_3^{\widehat{b}(9)-\widehat{b}(3)} \widehat{x}_3^{p-1} d(\widehat{x}_3) \quad \bmod \left(v_2^{\widehat{a}(9)+2}\right).$$

We will compute $d(\widehat{x}_k)$ modulo $(v_2^{\widehat{a}(k)+n(k)})$ inductively on $k$. Define integers $n(k)$ by

$$n(k) = \begin{cases} 2 & \text{for } k \equiv 3 \ (6), \\ 2p & \text{for } k \equiv 4 \ (6), \\ 2p^2 & \text{for } k \equiv 5 \ (6), \\ p^3 & \text{for } k \equiv 0 \ (6), \\ p^4 & \text{for } k \equiv 1 \ (6), \\ p^5 & \text{for } k \equiv 2 \ (6). \end{cases}$$

Instead of (6.1), we may define integers $\widehat{a}(k)$ for $k \geq 9$ inductively on $k$ by

$$\widehat{a}(k) = \begin{cases} p\widehat{a}(k-1) + p^3 + p^2 & \text{for } k \equiv 3 \bmod (6), \\ p\widehat{a}(k-1) + p^3 & \text{for } k \equiv 0 \bmod (6), \\ p\widehat{a}(k-1) & \text{for } k \not\equiv 0 \bmod (3). \end{cases}$$

We can prove this proposition in the same fashion as in the case $m = 4$. ∎

## 11. Proof of the main theorem

Here we will prove Theorem 6.4. First we consider the case that $m \geq 5$ (Theorem 1.3 included).

By Lemma 2.4 it suffices to show that the set

$$\left\{\delta\left(\widehat{x}_k^s/v_2^{\widehat{a}(k)}\right):\ k \geq 0 \text{ and } p \nmid s > 0\right\} \subset \mathrm{Ext}^1_{\Gamma(m+1)}\left(M_3^0\right)$$

is linearly independent over

$$R = \mathbf{Z}/(p)[v_3, v_3^{-1}, v_4, \cdots, v_{m+1}, v_{m+2}].$$

It follows from (2.2) that $\mathrm{Ext}^1_{\Gamma(m+1)}\left(M_3^0\right)$ is the free $\widehat{K}(3)_*$-module on the nine classes represented by

$$\left\{\widehat{t}_1, \widehat{t}_1^p, \widehat{t}_1^{p^2}, \widehat{t}_2, \widehat{t}_2^p, \widehat{t}_2^{p^2}, \widehat{t}_3, \widehat{t}_3^p, \widehat{t}_3^{p^2}\right\}. \tag{11.1}$$

We shall change this basis into another one. In $\Gamma(m+1)/I_3$ we have

$$\begin{aligned}
\eta_R(\widehat{v}_4) &= \widehat{v}_4 + v_3\widehat{t}_1^{p^3} - v_3^{p\omega}\widehat{t}_1,\\
\eta_R(\widehat{v}_5) &= \widehat{v}_5 + v_4\widehat{t}_1^{p^4} + v_3\widehat{t}_2^{p^3} - v_4^{p\omega}\widehat{t}_1 - v_3^{p^2\omega}\widehat{t}_2,\\
\eta_R(\widehat{v}_6) &= \widehat{v}_6 + v_5\widehat{t}_1^{p^5} + v_4\widehat{t}_2^{p^4} + v_3\widehat{t}_3^{p^3} - v_5^{p\omega}\widehat{t}_1 - v_4^{p^2\omega}\widehat{t}_2 - v_3^{p^3\omega}\widehat{t}_3.
\end{aligned}$$

so in $\mathrm{Ext}^1_{\Gamma(m+1)}(M_3^0)$ we have

$$\begin{aligned}
\widehat{t}_1^{p^3} &= v_3^{p\omega-1}\widehat{t}_1,\\
\widehat{t}_2^{p^3} &= v_3^{p^2\omega-1}\widehat{t}_2 + v_3^{-1}\left(v_4^{p\omega}\widehat{t}_1 - v_4\widehat{t}_1^{p^4}\right),\\
\widehat{t}_3^{p^3} &= v_3^{p^3\omega-1}\widehat{t}_3 + v_3^{-1}\left(v_4^{p^2\omega}\widehat{t}_2 + v_5^{p\omega}\widehat{t}_1 - v_4\widehat{t}_2^{p^4} - v_5\widehat{t}_1^{p^5}\right).
\end{aligned}$$

This means that we can replace the $\widehat{K}(3)_*$-basis of $\mathrm{Ext}^1_{\Gamma(m+1)}\left(M_3^0\right)$ of (11.1) with

$$\left\{\widehat{t}_1^{p^2}, \widehat{t}_1^{p^3}, \widehat{t}_1^{p^4}, \widehat{t}_2^{p^4}, \widehat{t}_2^{p^5}, \widehat{t}_2^{p^6}, \widehat{t}_3^{p^6}, \widehat{t}_3^{p^7}, \widehat{t}_3^{p^8}\right\}, \tag{11.2}$$

so its basis over $R$ is

$$\left\{\widehat{v}_3^t\widehat{t}_1^{p^2}, \widehat{v}_3^t\widehat{t}_1^{p^3}, \widehat{v}_3^t\widehat{t}_1^{p^4}, \widehat{v}_3^t\widehat{t}_2^{p^4}, \widehat{v}_3^t\widehat{t}_2^{p^5}, \widehat{v}_3^t\widehat{t}_2^{p^6}, \widehat{v}_3^t\widehat{t}_3^{p^6}, \widehat{v}_3^t\widehat{t}_3^{p^7}, \widehat{v}_3^t\widehat{t}_3^{p^8} : t \geq 0\right\}.$$

Now define integers $\widehat{c}(k)$ for $k \geq 0$ by

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq 8,\\ (p-1)p^{k-4} + \widehat{c}(k-4) & \text{for } k \geq 9.\end{cases}$$

For $k > 4$, write $k = k_0 + 4k_1$ with $5 \leq k_0 \leq 8$. Then the above is equivalent to

$$\widehat{c}(k) = (p-1)p^{k_0}\left(\frac{p^{4k_1}-1}{p^4-1}\right). \tag{11.3}$$

Then Lemmas 4.3 and 6.3 imply that

$$d(\widehat{x}_k) \equiv \pm v_2^{\widehat{a}(k)} v_3^{\widehat{b}(k)} \widehat{v}_3^{\widehat{c}(k)} \begin{cases} \widehat{t}_1^{p^{2+k}} & \text{for } 0 \le k \le 2, \\ \widehat{t}_2^{p^{1+k}} & \text{for } 3 \le k \le 4, \\ \widehat{t}_2^{p^6} & \text{for } k > 4 \text{ and } k \equiv 1 \bmod (4)\,, \\ \widehat{t}_3^{p^6} & \text{for } k > 4 \text{ and } k \equiv 2 \bmod (4)\,, \\ \widehat{t}_3^{p^7} & \text{for } k > 4 \text{ and } k \equiv 3 \bmod (4)\,, \\ \widehat{t}_3^{p^8} & \text{for } k > 4 \text{ and } k \equiv 4 \bmod (4)\,. \end{cases}$$

mod $\left(v_2^{1+\widehat{a}(k)}\right)$. This and the multiplicative property of the right unit imply that

(11.4)

$$\delta\left(\frac{\widehat{x}_k^s}{v_2^{\widehat{a}(k)}}\right) = \pm s v_3^{\widehat{b}(k)} \widehat{v}_3^{(s-1)p^k+\widehat{c}(k)} \begin{cases} \widehat{t}_1^{p^{2+k}} & \text{for } 0 \le k \le 2, \\ \widehat{t}_2^{p^{1+k}} & \text{for } 3 \le k \le 4, \\ \widehat{t}_2^{p^6} & \text{for } k > 4 \text{ and } k \equiv 1 \bmod (4)\,, \\ \widehat{t}_3^{p^6} & \text{for } k > 4 \text{ and } k \equiv 2 \bmod (4)\,, \\ \widehat{t}_3^{p^7} & \text{for } k > 4 \text{ and } k \equiv 3 \bmod (4)\,, \\ \widehat{t}_3^{p^8} & \text{for } k > 4 \text{ and } k \equiv 4 \bmod (4)\,. \end{cases}$$

In order to show that these elements (with $k \ge 0$ and $s > 0$ not divisible by $p$) are linearly independent over $R$, we may consider the exponents of $\widehat{v}_3$ above only for $k > 4$, because the ones for $0 \le k \le 4$ are clearly independent of those for $k > 0$, i.e., to show that the set

(11.5) $$\left\{\widehat{v}_3^{(s-1)p^k+\widehat{c}(k)} \colon k > 4 \text{ and } p \nmid s > 0\right\}$$

is linearly independent over $R$. For a fixed value of $k$, the exponents appearing in (11.5) are congruent to $\widehat{c}(k)$ modulo $(p^k)$ but not congruent (since $p \nmid s$) to $-p^k + \widehat{c}(k)$ modulo $(p^{k+1})$.

Now (11.3) implies that for $k > 4$,

$$\widehat{c}(k) \equiv p^k - \frac{(p-1)p^{k_0}}{p^4-1} \qquad \bmod \left(p^{k+1}\right);$$

so

$$(s-1)p^k + \widehat{c}(k) \equiv sp^k - \frac{(p-1)p^{k_0}}{p^4-1} \qquad \bmod \left(p^{k+1}\right).$$

Hence our condition is that the exponents associated with $k$ are congruent to $-\frac{(p-1)p^{k_0}}{p^4-1}$ modulo $(p^k)$ but not modulo $(p^{k+1})$, and these conditions are mutually exclusive for differing $k$. $\square$

*Proof for* $2 \le m \le 4$. The argument is the same, subject to the following changes. The integers $\widehat{c}(k)$ are defined by

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \le k \le 8, \\ (p-1)p^{k-6} + \widehat{c}(k-6) & \text{for } k \ge 9. \end{cases}$$

For $k > 2$, write $k = k_0 + 6k_1$ with $3 \le k_0 \le 8$. Then the above is equivalent to

$$\widehat{c}(k) = (p-1)p^{k_0}\left(\frac{p^{6k_1}-1}{p^6-1}\right).$$

Then (11.4) gets replaced by

$$\delta\left(\frac{\widehat{x}_k^s}{v_2^{\widehat{a}(k)}}\right) = \pm s v_3^{\widehat{b}(k)} \widehat{v}_3^{(s-1)p^k+\widehat{c}(k)} \begin{cases} \widehat{t}_1^{p^{2+k}} & \text{for } 0 \le k \le 2, \\ \widehat{t}_2^{p^{1+k_0}} & \text{for } 3 \le k_0 \le 5, \\ \widehat{t}_3^{p^{-2+k_0}} & \text{for } 6 \le k_0 \le 8. \end{cases}$$

(Notice that there is another term for $m = 3$ and $6 \le k \le 8$. We can ignore it, because it is linearly independent with the above coboundary.) Replacing the $\widehat{K}(3)$-basis of $\mathrm{Ext}_{\Gamma(m+1)}(M_3^0)$ by (11.2) for $m = 4$ and by

$$\left\{\widehat{t}_1^{p^2}, \widehat{t}_1^{p^3}, \widehat{t}_1^{p^4}, \widehat{t}_2^{p^4}, \widehat{t}_2^{p^5}, \widehat{t}_2^{p^6}, \widehat{t}_3^{p^4}, \widehat{t}_3^{p^5}, \widehat{t}_3^{p^6}\right\}$$

for $2 \le m \le 3$, we can argue for linear independence as before.

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